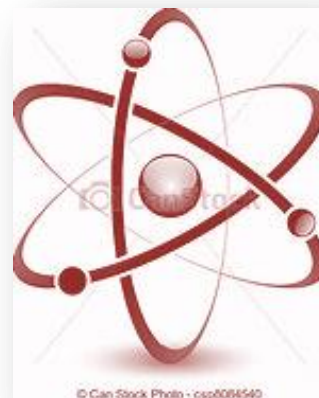

NEGATIVE EXPONENTS

The mass of a proton (the positively charged particle in the nucleus of every atom in the universe) is written in *scientific notation* as

$$1.67 \times 10^{-27} \text{ kilograms}$$

How do we interpret that? Is that a really big or a really small number? In this chapter, we will learn exactly what a negative exponent means.



❑ REVIEW OF THE FIVE LAWS OF EXPONENTS

Before we begin the discussion of negative exponents, it would be beneficial for us to review the meaning of an exponent and the Five Laws of exponents:

If n is a natural number ($n = 1, 2, 3, \dots$), then we know that the meaning of x^n is based upon repeated multiplication:

$$x^n = \underbrace{(x)(x)(x)\cdots(x)}_{n \text{ factors of } x}$$

For instance, we know that x^3 means $x \cdot x \cdot x$. We also learned that $x^0 = 1$ (as long as x itself isn't 0). But what does something like x^{-3} mean?

Now we recall the Five Laws of Exponents. The order doesn't make any difference, but I personally refer to the first one as the First Law of Exponents.

$$\begin{array}{lll}
 x^a x^b = x^{a+b} & \frac{x^a}{x^b} = x^{a-b} & (x^a)^b = x^{ab} \\
 (xy)^a = x^a y^a & \left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} &
 \end{array}$$

□ **DEVELOPING THE MEANING OF A NEGATIVE EXPONENT**

Consider the following power of x :

$$x^{-3}$$

What could it mean? To find out, let's multiply it by x^3 and see what happens:

$$x^{-3} \cdot x^3 = x^{-3+3} = x^0 = 1$$

which implies that

$$x^{-3} \cdot x^3 = 1$$

Now, x^{-3} is what we're analyzing, so let's "solve" for it (or isolate it) by dividing each side of the equation by x^3 :

$$\frac{x^{-3} \cdot x^3}{x^3} = \frac{1}{x^3}$$

$$\Rightarrow \boxed{x^{-3} = \frac{1}{x^3}}$$

We can do the same thing in general. Consider x^{-n} , where n is any number at all:

$$x^{-n} \cdot x^n = x^{-n+n} = x^0 = 1 \quad \Rightarrow \quad x^{-n} = \frac{1}{x^n}$$

In short,

$$x^{-n} = \frac{1}{x^n}$$

for ANY number n ,
assuming $x \neq 0$.

This should convince you that a negative exponent means **reciprocal**.

In fact, the exponent doesn't have to be a negative whole number.

[Even things like $w^{-5/4}$ and $h^{-\sqrt{2\pi}}$ represent reciprocals; in fact,
 $w^{-5/4} = \frac{1}{w^{5/4}}$ and $h^{-\sqrt{2\pi}} = \frac{1}{h^{\sqrt{2\pi}}}$.]

This fact, together with our knowledge of fractions and the Laws of Exponents, is all we need to understand the examples in this chapter.

EXAMPLE 1:

A. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

B. $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

C. $-3^{-4} = -\frac{1}{3^4} = -\frac{1}{81}$

D. $(-3)^{-4} = \frac{1}{(-3)^4} = \frac{1}{81}$

E. $\frac{1}{2^{-5}} = \frac{1}{\frac{1}{2^5}} = \frac{1}{\frac{1}{32}} = \frac{1}{1} \cdot \frac{32}{1} = 32$

See the difference?

$$F. \quad \frac{1}{10^{-3}} = \frac{1}{\frac{1}{10^3}} = \frac{1}{1} \cdot \frac{10^3}{1} = 10^3 = 1000$$

$$G. \quad \left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^3} = \frac{1}{\frac{8}{27}} = \frac{1}{1} \cdot \frac{27}{8} = \frac{27}{8}$$

$$H. \quad 0^{-5} = \frac{1}{0^5} = \frac{1}{0} = \text{Undefined}$$

Homework

1. Evaluate each expression:

$$a. \ 3^{-2} \quad b. \ 2^{-5} \quad c. \ 10^{-3} \quad d. \ 5^{-4} \quad e. \ 1^{-10}$$

$$f. \ -2^{-3} \quad g. \ (-3)^{-3} \quad h. \ -10^{-4} \quad i. \ (-5)^{-1} \quad j. \ -6^{-2}$$

$$k. \ \frac{1}{3^{-2}} \quad l. \ \frac{1}{10^{-4}} \quad m. \ \frac{1}{5^{-3}} \quad n. \ \left(\frac{3}{4}\right)^{-2} \quad o. \ \left(\frac{1}{6}\right)^{-3}$$

2. Explain why $x = 0$ is NOT allowed in the expression x^{-8} .

EXAMPLE 2:

$$A. \quad n^{-1} = \frac{1}{n}$$

$$B. \quad a^{-12} = \frac{1}{a^{12}}$$

$$C. \frac{x}{y^{-3}} = \frac{x}{\frac{1}{y^3}} = x \cdot \frac{y^3}{1} = xy^3$$

$$D. \frac{x^{-7}}{a^{12}} = \frac{\frac{1}{x^7}}{a^{12}} = \frac{1}{x^7} \cdot \frac{1}{a^{12}} = \frac{1}{a^{12}x^7}$$

$$E. a^2b^{-3} = a^2 \cdot \frac{1}{b^3} = \frac{a^2}{b^3}$$

$$F. R^{-1}T^{-7} = \frac{1}{R} \cdot \frac{1}{T^7} = \frac{1}{RT^7}$$

EXAMPLE 3:

$$A. x + y^{-5} = x + \frac{1}{y^5} = \underbrace{\left[\frac{y^5}{y^5} \right]}_{\text{to make a common denominator}} \frac{x}{1} + \frac{1}{y^5} = \frac{xy^5}{y^5} + \frac{1}{y^5} = \frac{xy^5 + 1}{y^5}$$

$$B. a^{-2} + b^{-7} = \frac{1}{a^2} + \frac{1}{b^7} = \frac{1}{a^2} \left[\frac{b^7}{b^7} \right] + \frac{1}{b^7} \left[\frac{a^2}{a^2} \right] = \frac{b^7 + a^2}{a^2b^7}$$

Homework

3. Simplify each expression (no negative exponents in the answer):

a. x^{-9}

b. a^{-30}

c. $\frac{a}{b^{-2}}$

d. $\frac{x^{-4}}{y^2}$

e. $\frac{w^3}{x^{-3}}$

f. $\frac{m^{-3}}{n^{-4}}$

g. $a^{-4}b^{-5}$

h. k^3p^{-5}

$$\begin{array}{llll} \text{i. } h^{-5}z^7 & \text{j. } \frac{w^{-5}}{w^{-5}} & \text{k. } x + y^{-1} & \text{l. } a^{-3} + b^2 \\ \text{m. } p^{-1} + r^{-1} & \text{n. } w^{-3} - z^{-2} & \text{o. } a^{-3} - a^{-3} & \text{p. } a^{-1}b^{-2}c^{-3} \end{array}$$

EXAMPLE 4: Using the Five Laws of Exponents:

A. $x^{-6}x^{-8} = x^{-14} = \frac{1}{x^{14}}$

Add the exponents

B. $(N^{-5})^{-7} = N^{35}$

Multiply the exponents

C. $(abc)^{-4} = a^{-4}b^{-4}c^{-4} = \frac{1}{a^4} \cdot \frac{1}{b^4} \cdot \frac{1}{c^4} = \frac{1}{a^4b^4c^4}$

Apply exponent to
each factor

D. $\frac{w^5}{w^{15}} = w^{5-15} = w^{-10} = \frac{1}{w^{10}}$

Subtract the exponents

E. $\frac{x^{-17}}{x^{-12}} = x^{-17-(-12)} = x^{-17+12} = x^{-5} = \frac{1}{x^5}$

Subtract the exponents

F. $\left(\frac{a}{b}\right)^{-3} = \frac{a^{-3}}{b^{-3}} = \frac{\frac{1}{a^3}}{\frac{1}{b^3}} = \frac{1}{a^3} \cdot \frac{b^3}{1} = \frac{b^3}{a^3}$

Apply exponent to
top and bottom

Another way: $\left(\frac{a}{b}\right)^{-3} = \frac{1}{\left(\frac{a}{b}\right)^3} = \frac{1}{\frac{a^3}{b^3}} = \frac{b^3}{a^3}$

6. $(x + y)^{-2}$

Be careful; no law of exponents applies here since the quantity $x + y$ consists of two terms — it is not a product — so do NOT apply the exponent to each term of the sum. Begin the problem by dealing with the meaning of the negative exponent.

$$(x + y)^{-2} = \frac{1}{(x + y)^2} = \frac{1}{(x + y)(x + y)} = \frac{1}{x^2 + 2xy + y^2}$$

Homework

4. Simplify each expression (no negative exponents in the answer):

a. $a^{-3}a^7$ b. $x^{-5}x^{-10}$ c. $k^{-10}k^{10}$ d. $m^{-5}w^{-3}$

e. $(x^2)^{-5}$ f. $(a^{-5})^3$ g. $(c^{-3})^{-4}$ h. $(b^2)^{-1}$

i. $(ax)^{-5}$ j. $(xyz)^{-3}$ k. $(x + y)^{-1}$ l. $(a - b)^{-2}$

m. $\frac{x^{-5}}{x^3}$ n. $\frac{z^5}{z^{-4}}$ o. $\frac{a^{-5}}{a^{-8}}$ p. $\frac{y^{-12}}{y^{-7}}$

q. $\left(\frac{a}{b}\right)^{-4}$ r. $\left(\frac{y}{x}\right)^{-1}$ s. $\left(\frac{a}{b + c}\right)^{-2}$ t. $\left(\frac{a + b}{xy + wz}\right)^0$

EXAMPLE 5: **Simplify each expression:**

$$A. \quad (2x^{-8})(-5x^6) = (2)(-5)x^{-8}x^6 = -10x^{-2} = -10\left(\frac{1}{x^2}\right) = -\frac{10}{x^2}$$

B. $(x^2y^{-3})^4 = (x^2)^4(y^{-3})^4 = x^8y^{-12} = x^8\left(\frac{1}{y^{12}}\right) = \frac{x^8}{y^{12}}$

$$c. \quad \left(\frac{a^{-3}}{b^4}\right)^{-5} = \frac{\left(a^{-3}\right)^{-5}}{\left(b^4\right)^{-5}} = \frac{a^{15}}{b^{-20}} = \frac{a^{15}}{\frac{1}{b^{20}}} = \frac{a^{15}}{1} \cdot \frac{b^{20}}{1} = a^{15}b^{20}$$

Final Note: In the Introduction, we learned about the mass of a proton. We're now in a position to see just what that number is:

[illegible]

[If available, check out the chapter on *Scientific Notation* for all the details and shortcuts regarding numbers like 1.67×10^{-27} .]

A proton certainly doesn't weigh very much! In addition, I hope you can appreciate why we use scientific notation (1.67×10^{-27}) rather than the number in the box.

Homework

5. Simplify each expression (no negative exponents in the answer):

a. $(2x^{-4})(3x^7)$	b. $(-3y^{-3})(2y^{-5})$	c. $(-4a^3)(a^{-7})$
d. $(u^3u^{-5})^{-2}$	e. $(w^{-5}w^{-1})^7$	f. $(a^{-2}b^{-3})^{-5}$
g. $(t^3u^{-4})^3$	h. $\left(\frac{x^2}{x^5}\right)^{-2}$	i. $\left(\frac{y^{-3}}{y^{-4}}\right)^5$
j. $\left(\frac{a^{-2}}{b^5}\right)^{-3}$	k. $\left(\frac{c^3}{d^{-4}}\right)^{-5}$	l. $\left(\frac{a^3b^{-4}}{x^{-12}z^{-2}}\right)^0$

6. Express each scientific notation number as a regular number:

a. 2.3×10^7	b. 7.11×10^{-5}	c. 5.09×10^{-10}
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Review Problems

- | | | | |
|----|-----------|------------------------------------|------------------------------------|
| 7. | Evaluate: | a. -2^{-4} | b. $(-2)^{-4}$ |
| 8. | Evaluate: | a. $\left(\frac{4}{5}\right)^{-3}$ | b. $\left(\frac{1}{9}\right)^{-1}$ |
| 9. | Simplify: | a. $\frac{a^{-2}}{b^{-3}}$ | b. $\frac{x^{-9}}{x^9}$ |

10. Simplify: a. $a^{-2}a^{-3}$ b. $a^{-2} - a^{-3}$
11. Simplify: a. $(x^{-10})^5$ b. $(z^{-5})^{-5}$
12. Simplify: $(3x^{-3}y^{-2})^{-1}(-4x^3y^{10})^{-2}$
13. Simplify: a. $\left(\frac{x^{-4}}{x^{-6}}\right)^{-2}$ b. $\left(\frac{a^3}{b^{-4}}\right)^5$
14. Simplify: a. $(uw)^{-2}$ b. $(u + w)^{-2}$
15. Explain why $x = 0$ is NOT allowed in the expression x^{-2} .
16. Express as a regular number: 4.9×10^{-13}

Solutions

1. a. $\frac{1}{9}$ b. $\frac{1}{32}$ c. $\frac{1}{1000}$ d. $\frac{1}{625}$ e. 1 f. $-\frac{1}{8}$
 g. $-\frac{1}{27}$ h. $-\frac{1}{10,000}$ i. $-\frac{1}{5}$ j. $-\frac{1}{36}$ k. 9 l. 10,000
 m. 125 n. $\frac{16}{9}$ o. 216
2. If x were 0 in the expression x^{-8} , we would have

$$x^{-8} = 0^{-8} = \frac{1}{0^8} = \frac{1}{0}, \text{ which is Undefined.}$$
3. a. $\frac{1}{x^9}$ b. $\frac{1}{a^{30}}$ c. ab^2 d. $\frac{1}{x^4y^2}$ e. x^3w^3 f. $\frac{n^4}{m^3}$
 g. $\frac{1}{a^4b^5}$ h. $\frac{k^3}{p^5}$ i. $\frac{z^7}{h^5}$ j. 1 k. $\frac{xy+1}{y}$ l. $\frac{1+a^3b^2}{a^3}$

$$\text{m. } \frac{r+p}{pr} \quad \text{n. } \frac{z^2-w^3}{w^3z^2} \quad \text{o. } 0 \quad \text{p. } \frac{1}{ab^2c^3}$$

$$\begin{array}{lllll} \text{4. a. } a^4 & \text{b. } \frac{1}{x^{15}} & \text{c. } 1 & \text{d. } \frac{1}{m^5w^3} & \text{e. } \frac{1}{x^{10}} \\ \text{f. } \frac{1}{a^{15}} & \text{g. } c^{12} & \text{h. } \frac{1}{b^2} & \text{i. } \frac{1}{a^5x^5} & \text{j. } \frac{1}{x^3y^3z^3} \\ \text{k. } \frac{1}{x+y} & \text{l. } \frac{1}{a^2-2ab+b^2} & \text{m. } \frac{1}{x^8} & \text{n. } z^9 & \\ \text{o. } a^3 & \text{p. } \frac{1}{y^5} & \text{q. } \frac{b^4}{a^4} & \text{r. } \frac{x}{y} & \text{s. } \frac{b^2+2bc+c^2}{a^2} \\ \text{t. } 1 & & & & \end{array}$$

$$\begin{array}{llllll} \text{5. a. } 6x^3 & \text{b. } -\frac{6}{y^8} & \text{c. } -\frac{4}{a^4} & \text{d. } u^4 & \text{e. } \frac{1}{w^{42}} & \text{f. } a^{10}b^{15} \\ \text{g. } \frac{t^9}{u^{12}} & \text{h. } x^6 & \text{i. } y^5 & \text{j. } a^6b^{15} & \text{k. } \frac{1}{c^{15}d^{20}} & \text{l. } 1 \end{array}$$

$$\text{6. a. } 23,000,000 \quad \text{b. } 0.0000711 \quad \text{c. } 0.0000000000509$$

$$\text{7. a. } -\frac{1}{16} \quad \text{b. } \frac{1}{16} \quad \text{8. a. } \frac{125}{64} \quad \text{b. } 9$$

$$\text{9. a. } \frac{b^3}{a^2} \quad \text{b. } \frac{1}{x^{18}} \quad \text{10. a. } \frac{1}{a^5} \quad \text{b. } \frac{a-1}{a^3}$$

$$\text{11. a. } \frac{1}{x^{50}} \quad \text{b. } z^{25} \quad \text{12. } \frac{1}{48x^3y^{18}}$$

$$\text{13. a. } \frac{1}{x^4} \quad \text{b. } a^{15}b^{20} \quad \text{14. a. } \frac{1}{u^2w^2} \quad \text{b. } \frac{1}{u^2+2uw+w^2}$$

$$\text{15. Because } 0^{-2} = \frac{1}{0^2} = \frac{1}{0} \text{ which is Undefined.}$$

$$\text{16. } 0.00000000000049$$

“A life spent making mistakes is not only more honorable, but more useful than a life spent doing nothing.”

– *George Bernard Shaw*